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Qualitative Analysis of Brans-Dicke Universes with a Cosmological Constant

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Abstract

Solutions to flat space Friedmann-Robertson-Walker cosmologies in Brans-Dicke theory with a cosmological constant are investigated. The matter is modelled as a γ-law perfect fluid. The field equations are reduced from fourth order to second order through a change of variables, and the resulting two-dimensional system is analyzed using dynamical system theory. When the Brans-Dicke coupling constant is positive ($\omega > 0$), all initially expanding models approach exponential expansion at late times, regardless of the type of matter present. If $\omega < 0$, then a wide variety of qualitatively distinct models are present, including nonsingular "bounce" universes, "vacillating" universes and, in the special case of $\omega = -1$, models which approach stable Minkowski spacetime with an exponentially increasing scalar field at late times. Since power-law solutions do not exist, none of the models appear to offer any advantage over the standard deSitter solution of general relativity in achieving a graceful exit from inflation.

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I. INTRODUCTION

A recent renewal of interest in Brans-Dicke (BD) theory [1] can be traced to the discovery by La and Steinhardt that the use of BD theory in place of general relativity can ameliorate the exit problem of inflationary cosmology [2]. This is possible because the interaction of the BD scalar field with the metric slows the expansion from exponential to power-law. Although the original "extended inflation" scenario appears to have been ruled out by observational constraints [3], models which survive include those based on more general scalar-tensor theories, such as "hyperextended inflation" [4], and hybrid models which include both a first-order phase transition and a period of slow-roll [5]. The renewed interest in scalar-tensor gravitation has led to several recent investigations into the generation of exact solutions for cosmology in such theories [6], as well as to some qualitative studies of the models which result [7–12]. It has also been pointed out recently that an inflationary era may result directly from the dynamics of the scalar field, without any potential or cosmological constant being necessary [13].

In this paper we are concerned with the behavior of homogeneous and isotropic cosmological models in Brans-Dicke gravity, with the addition of a positive cosmological constant. This differs from standard extended inflationary scenarios in that the vacuum energy is decoupled from the scalar field. The goal is simply to analyze the cosmological models which such a theory gives rise to, with an emphasis on the question of whether a viable inflationary model might exist. Previously, similar analyses have been performed for the case $\Lambda \neq 0$ with no matter present [10], and for the case $\Lambda = 0$ with additional matter present [11,12]. The treatment will closely parallel that given in Ref. [12], and the interested reader is referred to that paper for more detail.

In Sec. II, it is shown that the field equations for this theory can be reduced to a two-dimensional dynamical system in the case of flat space. In Sec. III, the equilibrium points are found and the corresponding solutions are discussed. In Sec. IV we summarize the results.

II. THE FIELD EQUATIONS

In this section the field equations are reduced to a planar dynamical system through a change of variables. For notation and conventions, the reader is referred to the parallel treatment given in [12]. Generalizing the action for Brans-Dicke theory to include a nonzero cosmological constant, it may be written as

$$S_{BD} = \int d^4x\sqrt{-g}\left(-\phi[R-2\Lambda] + \omega\frac{\phi^{,\mu}\phi_{,\mu}}{\phi} + 16\pi\mathcal{L}_m\right). \tag{2.1}$$

Taking Λ and ω constant, and varying this action with respect to the metric and the scalar field, one finds that the nontrivial components of the field equations in a homogeneous and isotropic (FRW) spacetime are

$$\left(\frac{\dot{a}}{a} + \frac{\dot{\phi}}{2\phi}\right)^2 + \frac{k}{a^2} = \left(\frac{2\omega+3}{12}\right)\left(\frac{\dot{\phi}}{\phi}\right)^2 + \frac{8\pi\rho}{3\phi} + \frac{\Lambda}{3}, \tag{2.2}$$

$$-\frac{1}{a^3}\frac{d}{dt}(\dot{\phi}a^3) = \left(\frac{8\pi}{3+2\omega}\right)\left(T^{\mu}{}_{\mu} - \frac{\Lambda\phi}{4\pi}\right), \tag{2.3}$$

where $a(t)$ is the cosmic scale factor. Assuming a perfect fluid form for the stress-energy tensor, *i.e.*, $T_{\mu\nu} = \mathrm{diag}(\rho, p, p, p)$, the usual conservation equation is also satisfied (the zeroth component of $T^{\mu\nu}{}_{;\nu} = 0$):

$$\dot{\rho} = -3\frac{\dot{a}}{a}(p+\rho). \tag{2.4}$$

Assuming only that $\rho > 0$ and $\phi > 0$, inspection of Eq. (2.2) reveals that we must have $\omega \geq -3/2$ in order to satisfy the field equations for all values of k and Λ. Furthermore, we see from the form of the action in Eq. (2.1) that the integrity of the theory is lost if $\omega = 0$. We therefore take $\omega \geq -3/2$ and $\omega \neq 0$ in what follows.

Now we take $k = 0$, and transform the fourth-order system specified by Eqs. (2.2-2.4) into a pair of coupled second-order equations in which, however, only first derivatives of the new variables appear. It will be sufficient for our purposes to consider only the models with $k = 0$, as any candidate for a viable inflationary model must at the very least solve the flatness problem. First, define the new variables

$$\Theta \equiv \left(\frac{\dot{a}}{a} + \frac{\dot{\phi}}{2\phi}\right), \tag{2.5}$$

$$\Sigma \equiv A\frac{\dot{\phi}}{\phi}, \tag{2.6}$$

where dots represent derivatives with respect to time, and $A \equiv (2\omega + 3)/12$. Next, parametrize the equation of state by writing $p = (\gamma - 1)\rho$, where, for example,

$$\gamma = \begin{cases} 0, & \text{false-vacuum energy;} \\ 1, & \text{pressureless dust;} \\ 4/3, & \text{radiation;} \\ 2, & \text{“stiff” matter.} \end{cases} \tag{2.7}$$

Finally, take $k = 0$ and rewrite Eqs. (2.2-2.4) in terms of these new variables. Then straightforward differentiation and resubstitution lead to the equivalent field equations

$$\dot{\Sigma} = \Theta^2(1 - 3\gamma/4) - \frac{\Sigma^2}{2A}(1 - 3\gamma/2) - 3\Theta\Sigma - \frac{\Lambda}{6}(1 - 3\gamma/2), \tag{2.8}$$

$$\dot{\Theta} = 3\frac{\Sigma^2}{A}(\gamma/2 - 1) - \frac{3\gamma\Theta^2}{2} + \frac{\Theta\Sigma}{2A} + \frac{\Lambda\gamma}{2}. \tag{2.9}$$

Eqs. (2.8) and (2.9) constitute a planar dynamical system in the variables Θ and Σ, and are the desired results of this section.

III. THE EQUILIBRIUM POINTS

The equilibrium points of the dynamical system are obtained by setting $\dot{\Theta}$ and $\dot{\Sigma}$ equal to zero in Eqs. (2.8) and (2.9), and then solving the resulting equations for Θ and Σ. As each of these equations is a second order polynomial, we expect in general four equilibrium points. When $\dot{\Theta} = \dot{\Sigma} = 0$, this implies that $\dot{a}/a = \Theta - \Sigma/2A$ and $\dot{\phi}/\phi = \Sigma/A$ are both constants, so that

$$a(t) = a_0 \exp\left[\left(\Theta - \frac{\Sigma}{2A}\right) t\right], \tag{3.1}$$

$$\phi(t) = \phi_0 \exp\left(\frac{\Sigma t}{A}\right). \tag{3.2}$$

Therefore a fixed point in the Θ–Σ system represents deSitter spacetime, with the addition of an exponentially varying scalar field. In the special case $\Theta = \Sigma/2A$, the solution is not deSitter but rather Minkowski spacetime with a scalar field. Note also that only equilibrium points at finite values have been considered. The global picture, including the behavior of Θ and Σ at infinity, may be obtained by various compactification methods [14]. Such an analysis is not necessary for our purposes, however, as we are primarily interested in whether viable inflationary models exist in this theory. In particular, one sees by inspection of Eqs. (2.8) and (2.9) that the origin of the Θ–Σ plane can never be an equilibrium point when $\Lambda \neq 0$. This immediately rules out the possiblity of a stable power-law solution, since such a solution would appear in that plane as line of constant slope, and would asymptotically approach equilibrium at the origin.

Although solution curves span the entire Θ–Σ plane, the requirement $\rho > 0$, where ρ is the energy density of the perfect fluid matter, eliminates some regions on physical grounds. It follows from Eq. (2.2) with $k = 0$ that (assuming $\phi > 0$) any point (Θ_0, Σ_0) satisfying ${\Theta_0}^2 > \Sigma_0^2/A + \Lambda/3$ will lie in $\rho > 0$, whereas points satisfying ${\Theta_0}^2 < \Sigma_0^2/A + \Lambda/3$ lie in $\rho < 0$ and thus do not represent physical solutions. Now let us proceed with the analysis. Although we restrict ourselves to the consideration of models with $\Lambda > 0$, it is clear that the techniques can easily be extended to models with a negative cosmological constant.

One pair of equilibrium points are always present regardless of the value of γ; they satisfy the field equations for $\rho = 0$ and are thus vacuum solutions. These points are

$$(\Theta_0, \Sigma_0)_{1,2} = \pm\left[\frac{\Lambda(2\omega+3)}{2(3\omega+4)}\right]^{1/2}(1, 1/6) \tag{3.3}$$

and they represent the solutions

$$a(t) = a_0 \exp\left\{\pm(\omega+1)\left[\frac{2\Lambda}{(2\omega+3)(3\omega+4)}\right]^{1/2} t\right\}, \tag{3.4}$$

$$\phi(t) = \phi_0 \exp\left\{\pm\left[\frac{2\Lambda}{(2\omega+3)(3\omega+4)}\right]^{1/2} t\right\}, \tag{3.5}$$

These solutions have previously been noted in the literature [7,10], and are attractors for most, but not all, of the initially expanding models in this theory (*cf.* discussion below). Note that if $\omega = -1$, then there are solutions where the geometry is Minkowski and the scalar field either grows or shrinks exponentially. Under the field redefinition $\phi \to e^{-\Phi}$, these can be identified as the static "linear dilaton" solutions of string cosmology [15].

The location and stability of the remaining equilibrium points depends upon the value of γ, as well as upon ω and Λ. Hence it is convenient to classify the models, and to discuss the overall character and stability of the solutions, according to the equation of state of the perfect fluid. The additional equilibrium points are as follows:

$$(\gamma = 0): \quad (\Theta_0, \Sigma_0)_{3,4} = \pm \left[\frac{\Lambda}{6}\right]^{1/2} (1, 0) \tag{3.6}$$

$$(\gamma = 1): \quad (\Theta_0, \Sigma_0)_{3,4} = \pm \left[\frac{\Lambda}{6(3\omega + 4)}\right]^{1/2} \left(1, \frac{2\omega + 3}{2}\right) \tag{3.7}$$

$$(\gamma = 4/3): \quad (\Theta_0, \Sigma_0)_{3,4} = \pm \left[\frac{\Lambda}{2(2\omega + 3)}\right]^{1/2} \left(1, \frac{2\omega + 3}{3}\right) \tag{3.8}$$

$$(\gamma = 2): \quad (\Theta_0, \Sigma_0)_{3,4} = \pm \left[\frac{2\Lambda}{3}\right]^{1/2} \left(1, \frac{2\omega + 3}{4}\right) \tag{3.9}$$

The details of the stability analysis are given in Appendix A. Table 1 of that appendix lists the eigenvalues of the equilibrium points, and Table 2 explicity states the existence and stability of each equilibrium point as a function of ω. Note that in some cases, the solutions represented by a given equilibrium point require negative energy density, and are therefore physically uninteresting. In such cases we have simply written "$\rho < 0$" in the tables.

The overall character of the solutions may be further examined by numerically integrating the solutions $\Theta(t)$ and $\Sigma(t)$ for a variety of initial conditions with each qualitatively distinct set of parameters $(\Lambda, \omega, \gamma)$. Figures 1–4 show the results of this procedure, where we have selected $\Lambda = 3$ arbitrarily. The shaded regions in each case require $\rho < 0$, and so are disallowed physically. Curves to the right of the line $da/dt = 0$ represent expanding universes, and those to its left represent contracting universes. Note that if $-1 < \omega < 0$, as for example in Fig. 1b, then there exist nonsingular "bounce" models which pass smoothly from contraction to expansion, and "vacillating" models which pass from expansion to contraction to reexpansion, or the time reversal of this behavior. There also exist extreme cases of "vacillation"; some models can vacillate several times before settling down to exponential contraction at late times, as shown in Fig. 4d.

IV. SUMMARY

We have shown that in Brans-Dicke theory with a positive cosmological constant, a wide range of flat-space models exist, including some with no analogues in general relativity. In general, these models are parametrized by the initial conditions of the scalar field, the value of the BD coupling constant, an initial expansion rate, an equation of state for the matter and the value of the cosmological constant. The first two parameters are, of course, absent in general relativity. Most, but not all, of the expanding models asymptotically approach vacuum deSitter spacetime at late times. Power-law expansion is not possible when Λ is nonzero.

If $\omega > -1$, there exist two finite-valued equilibrium points of physical interest, representing the vacuum solutions (3.4) and (3.5). One of these is stable and corresponds to expanding deSitter spacetime, and all initially expanding models approach this solution asymptotically. As $\omega \to \infty$, the solution becomes identical to the deSitter universe where $a(t) \sim e^{\Lambda t/3}$ and $\phi(t) = \text{constant}$, in accordance with the well-established correspondence between GR and BD theory in this limit. When $\omega > 0$, all contracting models contract to a singularity; however, if $-1 < \omega < 0$, then nonsingular "bounce" models are also possible, and these may or may not approach deSitter spacetime at late times, depending upon the initial conditions of the model. This behavior has been noted by other authors [16]. Also there are "vacillating" models, which expand from a big bang, slow down, and recontract before continuing their expansion and approaching deSitter spacetime. The time-reversal of this behavior also exists.

If $\omega = -1$, there exist "static-exponential" solutions, where the geometry is Minkowski spacetime while the scalar field changes exponentially with time. The solution with increasing scalar is found to be stable, and all models which expand from a big bang approach it at late times, regardless of the type of matter present. The stability of these models may be explained by the fact that in the Newtonian limit, $\phi \sim G^{-1}$ [17]. Hence the exponential increase in the scalar field corresponds to an exponential weakening of the gravitational interactions, ensuring that the universe does not recollapse regardless of its matter content.

If $-3/2 < \omega < -1$, then the behavior of the models depends upon the type of matter present, and we can distinguish two classes of behavior. **(i)** In the cases of false-vacuum energy and pressureless dust, only the equilibrium points representing the vacuum solutions (3.4) and (3.5) are present in the regime $\rho > 0$, and the contracting solution is stable. All models which start from a big bang eventually contract to a singularity; this collapse will become asymptotically exponential if $-4/3 < \omega < -1$, or superexponential if $-3/2 < \omega < -4/3$. There also exist models which start with a finite rate of expansion and expand perpetually. **(ii)** In the cases of radiation and "stiff" matter, the dynamical system undergoes a qualitative change at a particular value of ω. This critical value is $\omega_c = -5/4$ for radiation, and $\omega_c = -7/6$ for "stiff" matter. If $\omega_c \leq \omega < -1$, then only the equilibrium points representing (3.4) and (3.5) are present with $\rho \geq 0$, and the contracting solution is stable. If $-4/3 < \omega < \omega_c$, then there are four equilibrium points in the regime $\rho \geq 0$, representing both vacuum and non-vacuum deSitter solutions; however, only the contracting non-vacuum solution is stable. In these solutions, the exponential growth of the scalar field exactly balances that of the energy density, so that the ratio ρ/ϕ is constant. Thus the ordinary matter acts exactly like a cosmological constant, since it is this ratio which appears as the matter source term in the field equation (2.2). All models will collapse to a singularity; the collapse will be asymptotically exponential for models starting from a big bang. If $-3/2 < \omega < -4/3$, the unstable vacuum equilibrium points no longer exist; otherwise the behavior of the models remains unchanged.

Although somewhat exotic, the models discussed here do not seem to have any hope of solving the graceful exit problem of inflationary cosmology. In models of extended inflation, the mediation of the scalar field slows the expansion from exponential to power-law, so that the Hubble parameter decreases with time and true-vacuum bubble nucleation may complete the inflation-ending phase transition. Here, however, the cosmological constant induces deSitter spacetime, and all of the problems of Guth's "old" inflation recur. In addition, one is faced with the question of the origin of the cosmological constant in these models. Although there is no *a priori* reason to exclude such a term from the field equations, the usual explanation of a field with a nonzero potential as the source of the vacuum energy is not available in this case, since such a potential term would be coupled to the scalar field.

V. ACKNOWLEDGEMENTS

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APPENDIX A:

Defining $\xi^{(1)} \equiv \Theta - \Theta_0$ and $\xi^{(2)} \equiv \Sigma - \Sigma_0$, the dynamical system specified by (2.8) and (2.9) may be written in the form

$$\frac{d\vec{\xi}}{dt} = \mathrm{J}\vec{\xi} \quad + \quad \ldots \quad , \tag{A1}$$

where the Jacobian is

$$\mathrm{J} = \begin{pmatrix} -3\gamma\Theta_0 + \Sigma_0/2A & -6(\gamma/2-1)\Sigma_0/A + \Theta_0/A \\ 2\Theta_0(1-3\gamma/4) - 3\Sigma_0 & -(1-3\gamma/2)\Sigma_0/A - 3\Theta_0 \end{pmatrix}. \tag{A2}$$

In cases where the eigenvalues of the Jacobian all have nonvanishing real part, the fixed point is called hyperbolic and we can determine its stability from the signs of those real parts: if the real part of each of the eigenvalues is negative at a given equilibrium point, the solution is stable at that point; if the real part of each eigenvalue is positive, or if the real part of one eigenvalue is positive and that of the other is negative, then the solution is unstable at that point. Finally, if the real part of any of the eigenvalues is zero at a point, then the point is called nonhyperbolic and its stability in the neighborhood of that point cannot be determined by this method [14].

Table 1 shows the eigenvalues of the Jacobian for each equilibrium point, and Table 2 explicitly states the existence and stability of the equilibrium points as a function of ω. In cases where the solution represented by the equilibrium point requires negative energy density, we have simply written "$\rho < 0$". The points are labelled in accordance with the conventions in the text, *i.e.,* for each value of γ, points 1 and 2 represent vacuum solutions, and points 3 and 4 represent non-vacuum solutions.

TABLES

TABLE I. Eigenvalues of the Jacobian Matrix for BD Cosmology with $\Lambda > 0$

Equilibrium Point	λ_1	λ_2
$(\gamma = 0)_1$	$-[2\Lambda/(2\omega+3)(3\omega+4)]^{1/2}$	$-[2\Lambda(3\omega+4)/(2\omega+3)]^{1/2}$
$(\gamma = 0)_2$	$+[2\Lambda(3\omega+4)/(2\omega+3)]^{1/2}$	$+[2\Lambda/(2\omega+3)(3\omega+4)]^{1/2}$
$(\gamma = 0)_3$	$\rho < 0$	$\rho < 0$
$(\gamma = 0)_4$	$\rho < 0$	$\rho < 0$
$(\gamma = 1)_1$	$-[2\Lambda(3\omega+4)/(2\omega+3)]^{1/2}$	$-[2\Lambda(3\omega+4)/(2\omega+3)]^{1/2}$
$(\gamma = 1)_2$	$+[2\Lambda(3\omega+4)/(2\omega+3)]^{1/2}$	$+[2\Lambda(3\omega+4)/(2\omega+3)]^{1/2}$
$(\gamma = 1)_3$	$\rho < 0$	$\rho < 0$
$(\gamma = 1)_4$	$\rho < 0$	$\rho < 0$
$(\gamma = 4/3)_1$	$-[2\Lambda(3\omega+4)/(2\omega+3)]^{1/2}$	$-(4\omega+5)[2\Lambda/(2\omega+3)(3\omega+4)]^{1/2}$
$(\gamma = 4/3)_2$	$+(4\omega+5)[2\Lambda/(2\omega+3)(3\omega+4)]^{1/2}$	$+[2\Lambda(3\omega+4)/(2\omega+3)]^{1/2}$
$(\gamma = 4/3)_3$	$C(-1+\sqrt{64\omega+81})$[a]	$C(-1-\sqrt{64\omega+81})$[a]
$(\gamma = 4/3)_4$	$C(+1+\sqrt{64\omega+81})$[a]	$C(+1-\sqrt{64\omega+81})$[a]
$(\gamma = 2)_1$	$-[2\Lambda(3\omega+4)/(2\omega+3)]^{1/2}$	$-(6\omega+7)[2\Lambda/(2\omega+3)(3\omega+4)]^{1/2}$
$(\gamma = 2)_2$	$+(6\omega+7)[2\Lambda/(2\omega+3)(3\omega+4)]^{1/2}$	$+[2\Lambda(3\omega+4)/(2\omega+3)]^{1/2}$
$(\gamma = 2)_3$	$C(-\sqrt{3(2\omega+3)}+\sqrt{102\omega+121})$[a]	$C(-\sqrt{3(2\omega+3)}-\sqrt{102\omega+121})$[a]
$(\gamma = 2)_4$	$C(+\sqrt{3(2\omega+3)}+\sqrt{102\omega+121})$[a]	$C(+\sqrt{3(2\omega+3)}-\sqrt{102\omega+121})$[a]

[a] $C \equiv [\Lambda/2(2\omega+3)]^{1/2}$

TABLE II. Existence and Stability of the Equilibrium Points

Equilibrium Point	$-3/2 < \omega \leq -4/3$	$-4/3 < \omega < \omega_c$[a]	$\omega \geq \omega_c$
$(\gamma = 0,\ 1)_1$	nonexistent	N/A	stable
$(\gamma = 0,\ 1)_2$	nonexistent	N/A	unstable
$(\gamma = 0,\ 1)_3$	$\rho < 0$	N/A	$\rho < 0$
$(\gamma = 0,\ 1)_4$	$\rho < 0$	N/A	$\rho < 0$
$(\gamma = 4/3,\ 2)_1$	nonexistent	unstable	stable
$(\gamma = 4/3,\ 2)_2$	nonexistent	unstable	unstable
$(\gamma = 4/3,\ 2)_3$	stable	stable	$\rho < 0$
$(\gamma = 4/3,\ 2)_4$	unstable	unstable	$\rho < 0$

[a] $\omega_c(\gamma = 0, 1) = -4/3$; $\omega_c(\gamma = 4/3) = -5/4$; $\omega_c(\gamma = 2) = -7/6$

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FIGURES

FIG. 1. The evolution of solutions in the Θ–Σ plane with $\gamma = 0$ (false-vacuum energy), where $\Lambda = 3$ has been selected for convenience. (a) $\omega = 9/2$, representative of $\omega > 0$. Solutions to the right of the line $da/dt = 0$ represent expanding universes; those to the left are contracting universes. The shaded region requires negative energy density and so is disallowed physically. (b) $\omega = -1/2$, representative of $-1 < \omega < 0$. The line $da/dt = 0$ has moved into the regime of positive energy density, so that some models pass smoothly from contraction to expansion, a general feature of models with $\omega < 0$. (c) $\omega = -1$. Note the attractive nature of the "static-exponential" solution. (d) $\omega = -1.4$, representative of $-3/2 < \omega < -4/3$. All models collapse to a singularity regardless of initial conditions.

FIG. 2. Some models with $\gamma = 1$ (pressureless dust) and $\Lambda = 3$. (a) $\omega = 9/2$, representative of $\omega > 0$. (b) $\omega = -1/2$, representative of $-1 < \omega < 0$. (c) $\omega = -1$. (d) $\omega = -1.4$, representative of $-3/2 < \omega < -4/3$.

FIG. 3. Some models with $\gamma = 4/3$ (radiation) and $\Lambda = 3$. (a) $\omega = -1/2$, representative of $-1 < \omega < 0$. (b) $\omega = -1$. (c) $\omega = -1.2$, representative of $-5/4 \leq \omega < -1$. (d) $\omega = -1.4$, representative of $-3/2 < \omega < -5/4$.

FIG. 4. Some models with $\gamma = 2$ ("stiff matter") and $\Lambda = 3$. (a) $\omega = -1/2$, representative of $-1 < \omega < 0$. (b) $\omega = -1$. (c) $\omega = -1.2$, representative of $-7/6 \leq \omega < -1$. The locations of the equilibrium points are marked with dots for clarity. (d) $\omega = -1.4$, representative of $-3/2 < \omega < -7/6$. Only one solution is shown to simplify the plot.

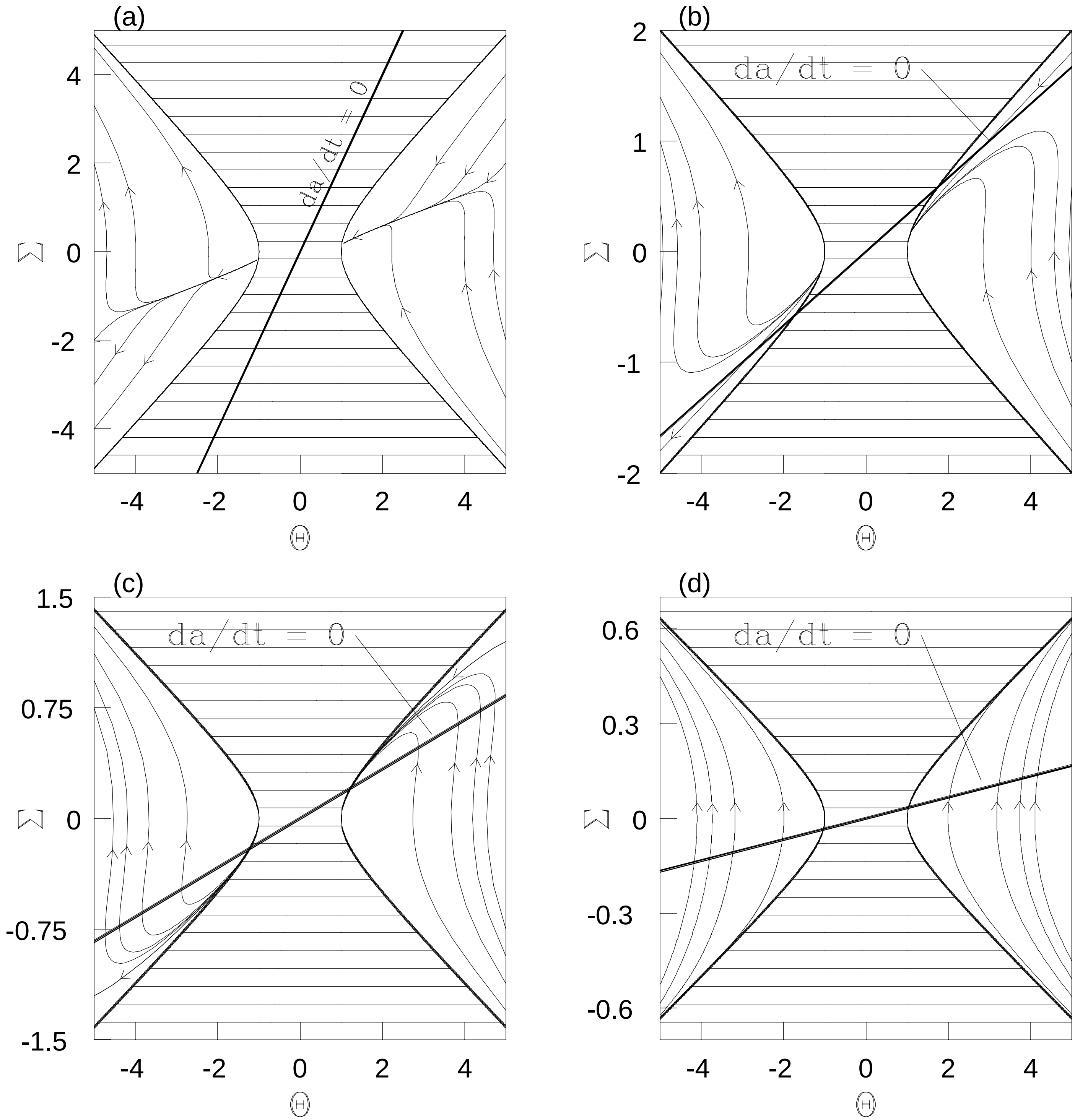


Fig. 1

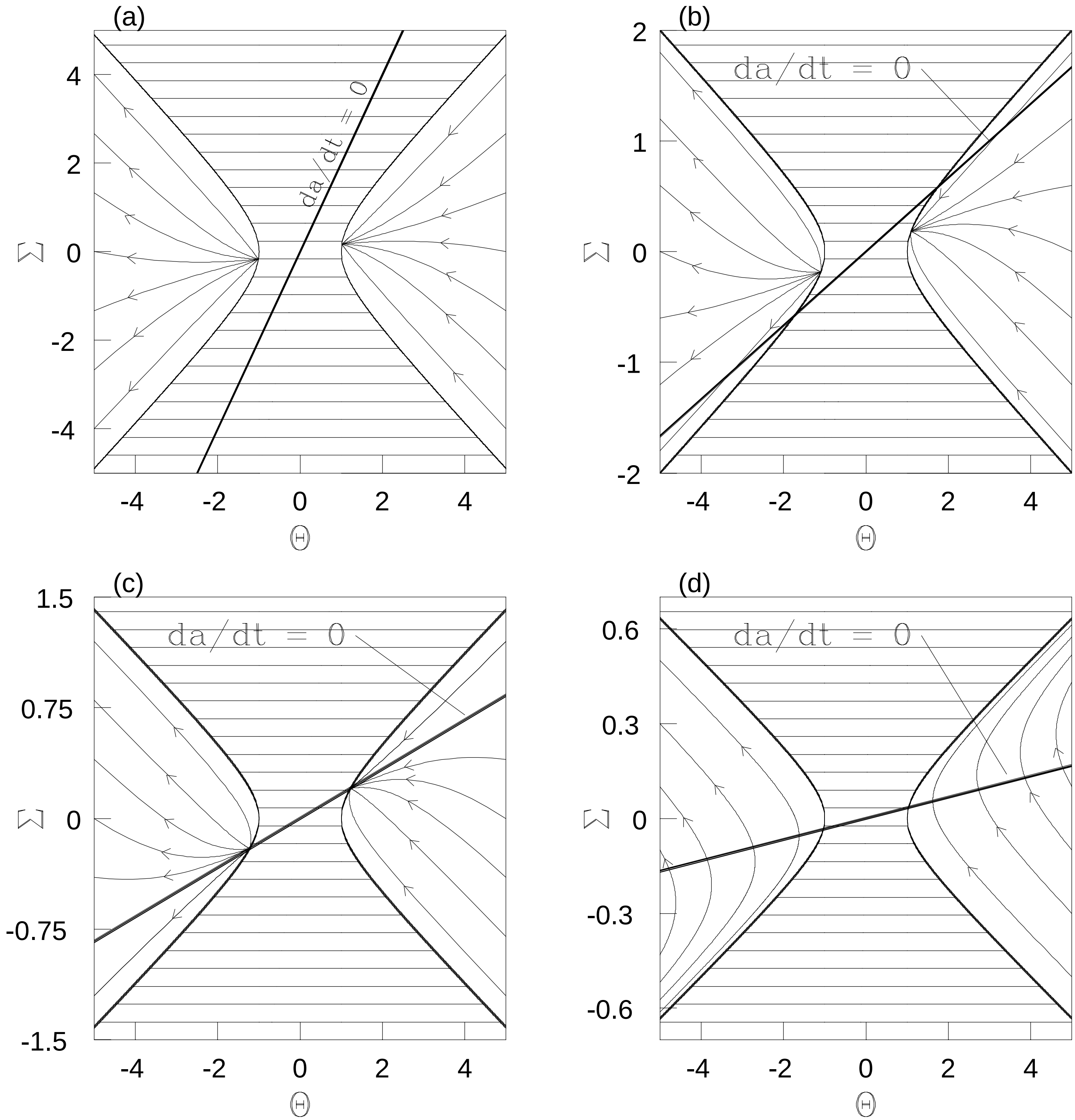
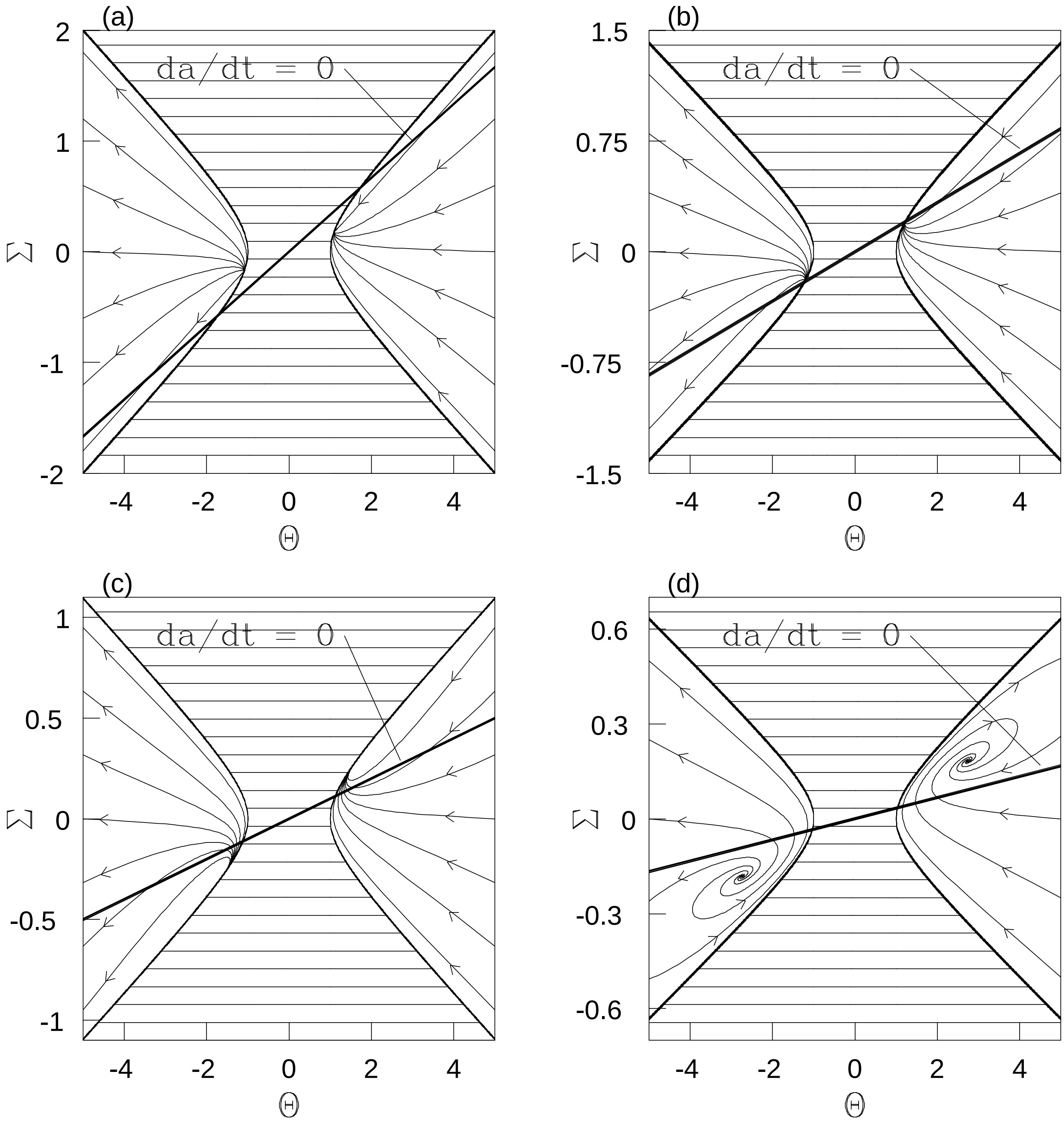
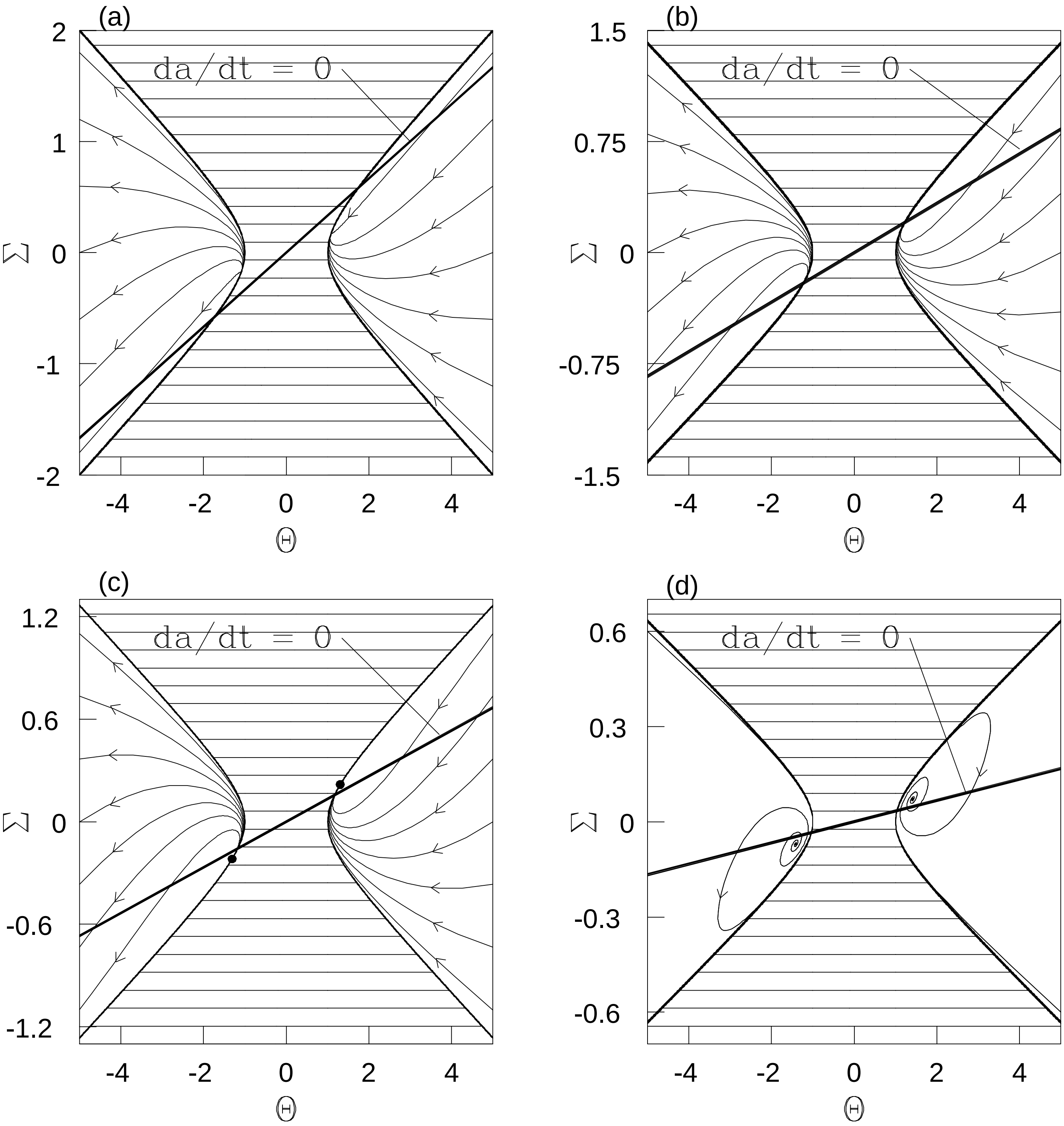


Fig. 2



(a)
da/dt = 0
2
1
0
-1
-2
-4
-2
0
2
4
Θ
Ξ
(b)
da/dt = 0
1.5
0.75
0
-0.75
-1.5
(c)
da/dt = 0
1
0.5
0
-0.5
-1
(d)
da/dt = 0
0.6
0.3
0
-0.3
-0.6

Fig. 3



(a)
da/dt = 0
2
1
0
-1
-2
-4
-2
0
2
4
Σ
Θ
(b)
da/dt = 0
1.5
0.75
0
-0.75
-1.5
-4
-2
0
2
4
Σ
Θ
(c)
da/dt = 0
1.2
0.6
0
-0.6
-1.2
-4
-2
0
2
4
Σ
Θ
(d)
da/dt = 0
0.6
0.3
0
-0.3
-0.6
-4
-2
0
2
4
Σ
Θ

Fig. 4